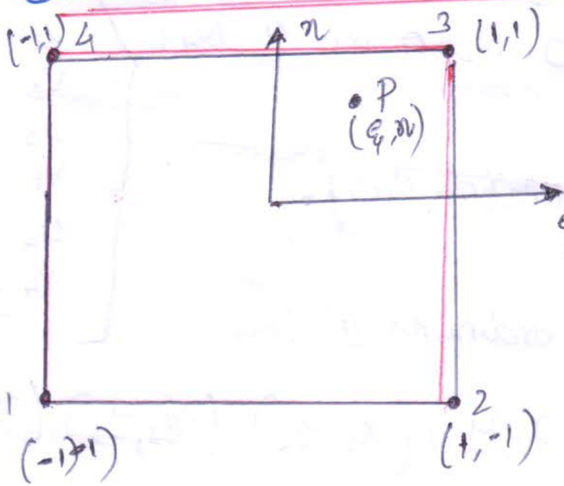
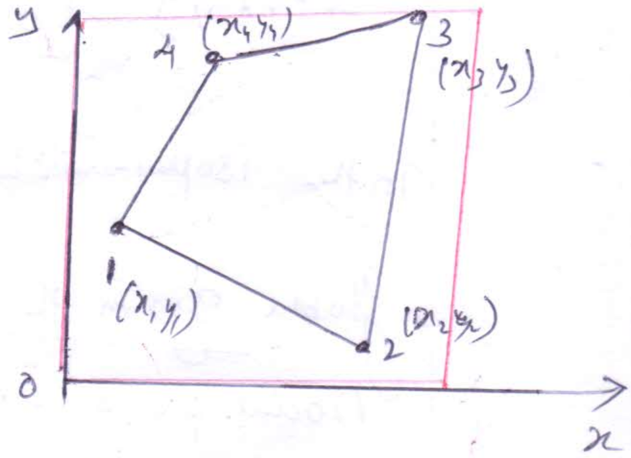
ELEMENT STIFENESS MATRIX Equation for 4 Noded Isoparametric Quadrilateral Element



(a) parent element



(b) Isoparametric element

Assembling element stiffness matrix for isoparametric element is a tedious process since it involves co-ordinate transformation from natural co-ordinate system to global co-ordinate system.

The displacement function u for parent rectangular element is given by

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \rightarrow (1)$$

The displacement function u for isoparametric quadrilateral element is given by

$$u = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix} \rightarrow (2)$$

We have to express the derivatives of a function in x, y. Co-ordinates interms of its derivatives in ε, η, co-ordinates. This can be done as follows.

Let $f = f(x, y)$,

$$f = [x(\varepsilon, \eta), y(\varepsilon, \eta)]$$

The relationship b/n natural co-ordinates and global co-ordinates can be calculated by using chain rule of partial differentiation:

$$\frac{\partial f}{\partial \varepsilon} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial \varepsilon} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \varepsilon} \rightarrow 3a$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \eta} \longrightarrow 3(b)$$

Arrange above eqn in matrix form:

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} \longrightarrow (4)$$

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} \leftarrow 4a$$

Where $[J]$ is the Jacobian matrix.

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

We know that

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4.$$

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi} \cdot x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} x_4 \rightarrow (5)$$

$$J_{12} = \frac{\partial y}{\partial \xi} = \frac{\partial N_1}{\partial \xi} y_1 + \frac{\partial N_2}{\partial \xi} y_2 + \frac{\partial N_3}{\partial \xi} y_3 + \frac{\partial N_4}{\partial \xi} y_4 \rightarrow (6)$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \frac{\partial N_1}{\partial \eta} x_1 + \frac{\partial N_2}{\partial \eta} x_2 + \frac{\partial N_3}{\partial \eta} x_3 + \frac{\partial N_4}{\partial \eta} x_4 \rightarrow (7)$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \frac{\partial N_1}{\partial \eta} y_1 + \frac{\partial N_2}{\partial \eta} y_2 + \frac{\partial N_3}{\partial \eta} y_3 + \frac{\partial N_4}{\partial \eta} y_4 \rightarrow (8)$$

We know that

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$\frac{\partial N_1}{\partial \xi} = \frac{1}{4}(-1)(1-\eta) = \frac{1}{4}(-)(1-\eta)$$

$$\frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1)(1-\eta) = \frac{1}{4}(1-\eta)$$

$$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1)(1+\eta) = \frac{1}{4}(1+\eta)$$

$$\frac{\partial N_4}{\partial \xi} = \frac{1}{4}(-1)(1+\eta) = \frac{1}{4}(-)(1+\eta)$$

$$\frac{\partial N_1}{\partial \eta} = \frac{1}{4}(1-\xi)(-1) = \frac{1}{4}(-)(1-\xi)$$

$$\frac{\partial N_2}{\partial \eta} = \frac{1}{4}(1+\xi)(-1) = \frac{1}{4}(-)(1+\xi)$$

$$\frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi)(1) = \frac{1}{4}(1+\xi)$$

$$\frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi)(1) = \frac{1}{4}(1-\xi).$$

$\rightarrow$ (I)

Substn the above Values in eqn (5), (6), (7), (8)

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$J_{21} = \frac{1}{4}\left[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\right]$$

$$J_{22} = \frac{1}{4}\left[-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4\right]$$

$\rightarrow$ (9)

from eqn. 4

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$= \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} \rightarrow 10$$

The strain displacement relations are given by.

$$e = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \rightarrow 11$$

Subst $f = u$. in 10

53

$$\begin{Bmatrix} \frac{du}{dx} \\ \frac{du}{dy} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \frac{du}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} \rightarrow (12)$$

lly

$$\begin{Bmatrix} \frac{dv}{dx} \\ \frac{dv}{dy} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} \rightarrow (13)$$

Comparing eqn (11), (12) and (13) we get

$$e = \begin{Bmatrix} \frac{du}{dx} \\ \frac{dv}{dy} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{du}{dx} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} \rightarrow (14)$$

We know that,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4.$$

We have,

$$\frac{du}{d\varepsilon} = \frac{\partial N_1}{\partial \varepsilon} u_1 + \frac{\partial N_2}{\partial \varepsilon} u_2 + \frac{\partial N_3}{\partial \varepsilon} u_3 + \frac{\partial N_4}{\partial \varepsilon} u_4.$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial N_1}{\partial \eta} u_1 + \frac{\partial N_2}{\partial \eta} u_2 + \frac{\partial N_3}{\partial \eta} u_3 + \frac{\partial N_4}{\partial \eta} \cdot u_4$$

$$\frac{\partial v}{\partial \varepsilon} = \frac{\partial N_1}{\partial \varepsilon} v_1 + \frac{\partial N_2}{\partial \varepsilon} v_2 + \frac{\partial N_3}{\partial \varepsilon} v_3 + \frac{\partial N_4}{\partial \varepsilon} v_4$$

$$\frac{\partial v}{\partial \eta} = \frac{\partial N_1}{\partial \eta} v_1 + \frac{\partial N_2}{\partial \eta} v_2 + \frac{\partial N_3}{\partial \eta} v_3 + \frac{\partial N_4}{\partial \eta} \cdot v_4.$$

assembly the above equation in matrix form. we get

$$\begin{Bmatrix} \frac{\partial u}{\partial \varepsilon} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \varepsilon} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & 0 & \frac{\partial N_3}{\partial \varepsilon} & 0 & \frac{\partial N_4}{\partial \varepsilon} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & 0 & \frac{\partial N_3}{\partial \varepsilon} & 0 & \frac{\partial N_4}{\partial \varepsilon} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \rightarrow 15$$

sub in 15 eqn. 14

$$\text{Strain } \{e\} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times$$

$$\times \begin{bmatrix} \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & \cdots & 0 \\ \cdots & & & & \\ \cdots & & & & \\ \cdots & & & & \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \longrightarrow 16$$

We know that

Strain $\{e\} = [B]\{u\}$ (17) from (I)

Compare (17) with (16) above eqn. And sub the value of pon... deri... wegs

$$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times$$

$$\begin{bmatrix} -(1+\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

This is a Strain displacement relationship matrix

$[B]$ Equation for isoparametric quadrilateral elements.

We know

$$[K] = \int_v [B]^T [D] [B] \cdot dV.$$

$$\Rightarrow [K] = t \iint [B]^T [D] [B] \cdot dx \cdot dy$$

$$[K] = t \int_{-1}^{1}\int_{-1}^{1} [B]^T \cdot [D] [B] \times |J| \times \partial\varepsilon \cdot \partial\eta.$$

$$(\because \partial x \cdot \partial y = |J|\, \partial\varepsilon\, \partial\eta)$$

where.

$t =$ thickness of element

$|J| \rightarrow$ determinant of the jacobian.

$\xi, \eta \rightarrow$ Natural Co-ordinates

$[B] \rightarrow$ strain displacement relation matrix,

$[D] \rightarrow$ Stress strain relationship matrix.

for 2D problems

Stress-strain relationship matrix

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \text{ (for plane stress condition)}$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \text{ (for plane strain conditions)}$$

Where $E \rightarrow$ Young's modulus.

μ = poisson ratio

Element Force Vector.

The element force vector is given by,

$$\{F\}_e = N^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

Where N is the shape function.

F_x is the load or force on x direction

F_y is a force on y direction.